Inequality for three non coplanar vectors.

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Let $x,y,z > 0$. Prove that

$$\sqrt{x^2-\sqrt{3}\,xy+y^2}+\sqrt{y^2-\sqrt{2}\,yz+z^2} \geq \sqrt{z^2-zx+x^2}.$$

Solution by Arkady Alt, San Jose, California, USA.

Let $\mathbf{x},\mathbf{y},\mathbf{z}$ be three non coplanar vectors in $\mathbb{R}^3$ such that $\|\mathbf{x}\| = x, \|\mathbf{y}\| = y, \|\mathbf{z}\| = z$ and $\widehat{\mathbf{x},\mathbf{y}} = 30^\circ, \widehat{\mathbf{y},\mathbf{z}} = 45^\circ, \widehat{\mathbf{z},\mathbf{x}} = 60^\circ$. Then $\sqrt{x^2-\sqrt{3}\,xy+y^2} = \sqrt{x^2-2\cos 30^\circ\cdot xy+y^2} = \|\mathbf{x}-\mathbf{y}\|, \sqrt{y^2-\sqrt{2}\,yz+z^2} = \sqrt{y^2-2\sqrt{2}\cos 45^\circ\cdot yz+z^2} = \|\mathbf{y}-\mathbf{z}\|, \sqrt{z^2-zx+x^2} = \sqrt{z^2-2\cos 60^\circ\cdot zx+x^2} = \|\mathbf{z}-\mathbf{x}\|$ and we have

$\|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}-\mathbf{z}\| \geq \|\mathbf{x}-\mathbf{y}+\mathbf{y}-\mathbf{z}\| = \|\mathbf{x}-\mathbf{z}\|.$